6 Problem Set 6

Problem 1.
Both schemes are calculated using

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)$$

We can see that the parameters given provides us with some simplified expressions, namely

$$f(\rho) = \rho(1-\rho) \qquad a_{i+\frac{1}{2}} = (1 - \rho_i - \rho_{i+1})$$

First, we examine Roe's scheme which provides a numerical flux of

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left[f(\rho_i) + f(\rho_{i+\frac{1}{2}})\right] - \frac{1}{2}\left|a_{i+\frac{1}{2}}\right|(\rho_{i+\frac{1}{2}} - \rho_i)$$

My simulation has the added parameters of having the traffic light at 0, with the range from -2 to 2, with a step size of 0.01, as the problem statement hints at a range of 400 0.01 steps. The initial condition is set appropriately, with a steady 0.8 density before the traffic light, and can be evaluated by appending a segment for this specific method

```
elseif method == "roe"
        for i = 1:m
            F[i] = 0.5*((rl[i]*(1-rl[i])) + (rr[i]*(1-rr[i]))) - 0.5*(abs(1-rl[i]-rr[i])*(rr[i] - rl[i]))

        end
```

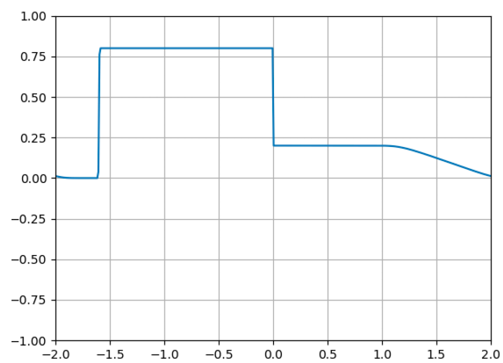
On the other hand, Godunov can be expressed by the piecewise flux function

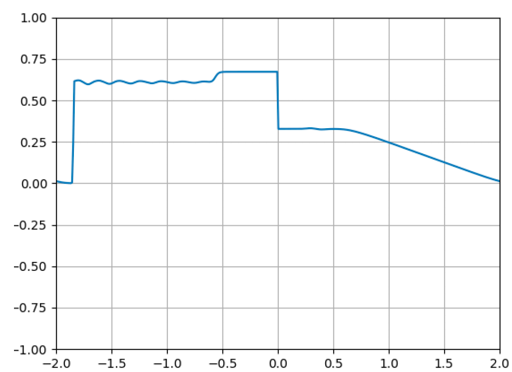
$$F_{i+\frac{1}{2}}^G = f(\rho(x_{+\frac{1}{2}}, t^{n+}))$$

and appending another method

```
if method == "godunov"
for i = 1:m
    if rl[i] > rr[i]
       F[i] = max(rl[i]*(1-rl[i]), rr[i]*(1-rr[i]))
    elseif rl[i] < rr[i]
       F[i] = min(rl[i]*(1-rl[i]), rr[i]*(1-rr[i]))
    else
         F[i] = 0
    end
```

testing for the boundary conditions which provide a consistent 0.8ρ at the section before the light provides





Note that Roe's scheme on the left does not take into account the shocks that Godunov's scheme does, on the right. This would be a result of the piecewise nature of the flux functions with Godunov.

Problem 2.
To get a node at $\frac{-\Delta x}{2}$ we shift everything over by 0.5 through rewriting

```
x = ((collect(1:m).-0.5) ./(m/4)).-2.0
```

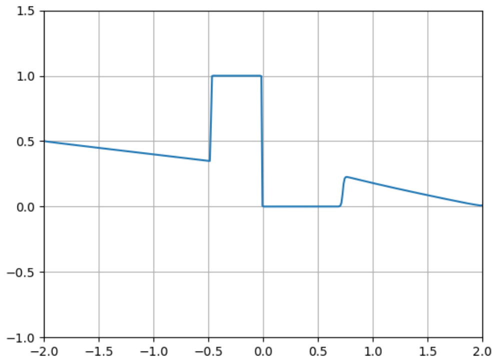
We can then find the node at which the traffic light is using

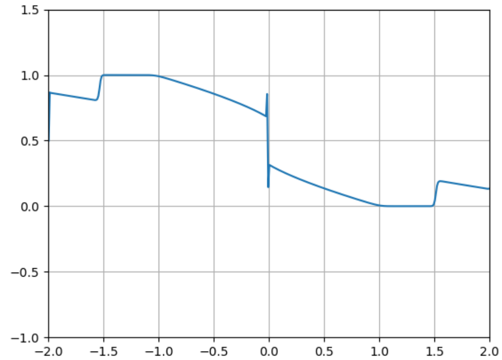
```
a = findmin(map(x->abs(x+(h/2)),x))[2]
```

and quite straightforwardly, can set the node at which the traffic light is to 0 when the light is red.

```
if div(it,125)%2 == 0
    F[a] = 0
end
```

Doing this, we observe the following results, the final time step is shown on the right





Also, we use the given formula

$$\dot{q} = \frac{1}{N^T}\sum_{n=1}^{N_T} f^n$$

to calculate the average flow of cars, until it no longer changes. This yields a value of 36.49201111461606.

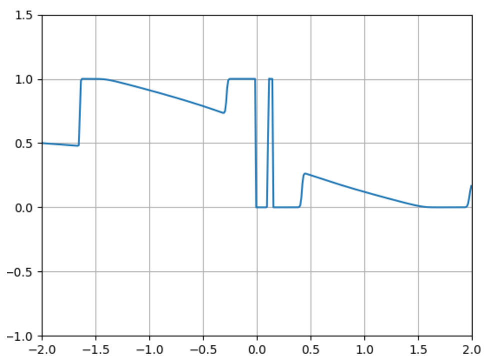
Problem 3.
We can find the node for 0.15 at the point given by

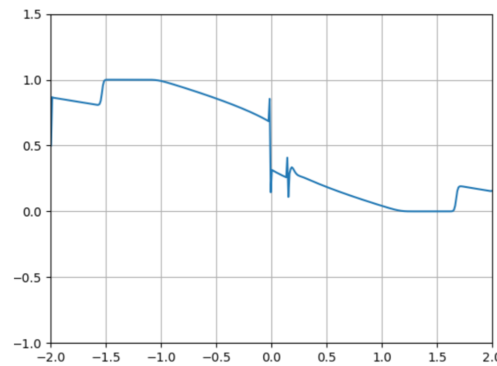
```
b = findmin(map(x->abs(x-0.15),x))[2]
```

and simply modify the condition for the traffic light by adding a second one, both with a period of $T = 1$

```
if div(it,125)%2 == 0
    F[a] = 0
end
if div(it,125)%2 == 0
    F[b] = 0
end
```

Note that the two if statements are to expedite the later part of the problem; so we can adjust the periods for both traffic lights. As our timestep is 0.008, we can adjust the delay τ by adding to the second if statement. No delay can be seen by the following, the right figure is at $T = 10$

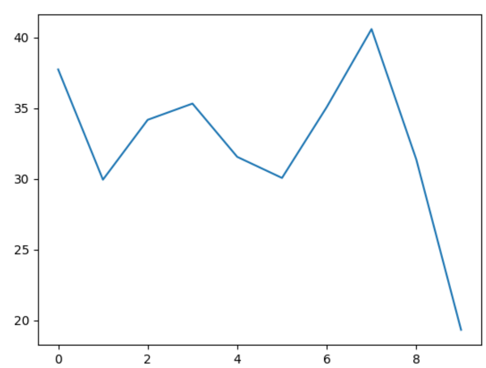




As each interval is given by 125 steps of 0.008 each, we can add 12.5 for each iteration of k. Running this 10 times with the respective values of k, and then evaluating each of the $\dot{q}$, we get the values

$$\begin{bmatrix} 36.49201111461606 \\ 37.75273450559253 \\ 29.955535036151215 \\ 34.189830035523464 \\ 35.337277520802274 \\ 31.568430577036786 \\ 30.07860504113374 \\ 35.09431340406473 \\ 40.607257371577106 \\ 31.383048080456778 \\ 19.333696130997627 \end{bmatrix}$$

for the average flow, or road capacity with different values, of k $0-9$ starting from the top. Graphing this out versus k values yields



Thus, we can see the optimal delay is given by $k = 7$ or $\tau = 0.7$ as it has the largest flow, 40.607257371577106.

UC Berkeley Math 228B, Spring 2019: Problem Set 6

Due April 25

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1}$$

with $\rho = \rho(x,t)$ the density of cars (vehicles/km), and $u = u(x,t)$ the velocity. Assume that the velocity u is given as a function of ρ:

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{2}$$

With $u_{\max}$ the maximum speed and $0 \le \rho \le \rho_{\max}$. The flux of cars is therefore given by:

$$f(\rho) = \rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{3}$$

We will solve this problem using a first order finite volume scheme:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right). \tag{4}$$

For the numerical flux function, we will consider two different schemes:

Roe's Scheme

The expression of the numerical flux is given by:

$$F^R_{i+\frac{1}{2}} = \frac{1}{2}\left[f(\rho_i) + f(\rho_{i+1})\right] - \frac{1}{2}\left|a_{i+\frac{1}{2}}\right|(\rho_{i+1} - \rho_i) \tag{5}$$

with

$$a_{i+\frac{1}{2}} = u_{\max}\left(1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\max}}\right). \tag{6}$$

Note that $a_{i+\frac{1}{2}}$ satisfies

$$f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}}(\rho_{i+1} - \rho_i). \tag{7}$$

Godunov's Scheme

In this case the numerical flux is given by:

$$F^G_{i+\frac{1}{2}} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}. \end{cases} \tag{8}$$

Turn page $\longrightarrow$

1. For both Roe's Scheme and Godunov's Scheme, look at the problem of a traffic light turning green at time $t = 0$. We are interested in the solution at $t = 2$ using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you get arises.

   Use the following problem parameters:

$$\begin{aligned} \rho_{\max} &= 1.0, \quad \rho_L = 0.8 \\ u_{\max} &= 1.0 \\ \Delta x &= \frac{4}{400}, \quad \Delta t = \frac{0.8\Delta x}{u_{\max}} \end{aligned} \tag{9}$$

   The initial condition at the instant when the traffic light turns green is

$$\rho(0) = \begin{cases} \rho_L, & x < 0 \\ 0, & x \geq 0 \end{cases} \tag{10}$$

   For problems 2 - 3, use only the scheme(s) which are valid models of the problem.

2. Simulate the effect of a traffic light at $x = -\frac{\Delta x}{2}$ which has a period of $T = T_1 + T_2 = 2$ units. Assume that the traffic light is $T_1 = 1$ units on red and $T_2 = 1$ units on green. Assume a sufficiently high flow density of cars (e.g. set $\rho = \frac{\rho_{\max}}{2}$ on the left boundary – giving a maximum flux), and determine the average flow, or capacity of cars over a time period T.

   The average flow can be approximated as

$$\dot{q} = \frac{1}{N_T} \sum_{n=1}^{N_T} f^n = \frac{1}{N^T} \sum_{n=1}^{N_T} \rho^n u^n, \tag{11}$$

   where N_T is the number of time steps for each period T. You should run your computation until $\dot{q}$ over a time period does not change. Note that by continuity $\dot{q}$ can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior domain).

   Note: A red traffic light can be modeled by simply setting $F_{i+\frac{1}{2}} = 0$ at the position where the traffic light is located.

3. Assume now that we simulate two traffic lights, one located at $x = 0$, and the other at $x = 0.15$, both with a period T. Calculate the road capacity (= average flow) for different delay factors. That is if the first light turns green at time t, then the second light will turn green at $t + \tau$. Solve for $\tau = k\frac{T}{10}$, $k = 0, \ldots, 9$. Plot your results of capacity vs τ and determine the optimal delay τ.